Math 142 Lecture 21 Notes

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1 Abelianization and Other Algebraic Topology Topics

Many thanks to Jiabao Yang, who provided me with his notes, since I missed this lecture.

1.1 Abelianization

Last lecture, we encountered a problem: groups can be complicated, and we can't tell whether they are different or not based on group presentations.[1] The solution, in our case, is that abelian groups are not as complicated.

Definition 1.1. If G is a group, then its *Abelianization* $\mathrm{Ab}(G)$ is G/N, where N is the smallest normal subgroup of G containing $g_1 g_2 g_1^{-1} g_2^{-1}$ for all $g_1, g_2 \in G$.

If $G = \langle a_1, \dots, a_n \mid r_1 = 1, \dots, r_m = 1 \rangle$, then we add $n(n-1)/2$ relations to get

$$\mathrm{Ab}(G) = \langle a_1, \dots, a_n \mid r_1 = 1, \dots, r_m = 1, a_1 a_2 = a_2 a_1, a_1 a_3 = a_3 a_1, \dots, a_{n-1} a_n = a_n a_{n-1} \rangle .$$

Example 1.1.

$$\mathrm{Ab}(F_2) = \mathrm{Ab}(\langle a_1, a_2 \rangle) = \langle a_1, a_2 \mid a_1 a_2 = a_2 a_1 \rangle \cong \mathbb{Z}^2.$$

Here is a fact we will not prove.

Theorem 1.1. *IF* $G \cong G'$*, then* $\mathrm{Ab}(G) \cong \mathrm{Ab}(G')$.

The converse is not true, however.

Example 1.2. $F_2 \not\cong \mathbb{Z}^2$, but $\mathrm{Ab}(F_2) \cong \mathbb{Z}^2 \cong \mathrm{Ab}(\mathbb{Z}^2)$.

Example 1.3. Let A_5 be the alternating group on five elements. This is nontrivial, but $\mathrm{Ab}(A_5) \cong 1$.

Here is another fact we will not prove.

[1] In general, this problem is undecidable.

Proposition 1.1. *If $r_i = 1$ is a relation in G, then any permutation of the letters of r_i is an equivalent relation in* $\text{Ab}(G)$.

Example 1.4. Let $G = \langle a, b \mid abab^{-1} = 1 \rangle$. Then

$$\begin{aligned}
\text{Ab}(G) &= \langle ab \mid abab^{-1} = 1, ab = ba \rangle \\
&= \langle ab \mid aabb^{-1} = 1, ab = ba \rangle \\
&= \langle ab \mid a^2 = 1, ab = ba \rangle \\
&\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}.
\end{aligned}$$

So $\text{Ab}(\langle a, b \mid abab^{-1} = 1 \rangle) \cong \text{Ab}(\langle a, b \mid a^2 = 1 \rangle)$.

So if we reorder the group before abelianization, we get the same group (up to isomorphism) after abelianization.

Example 1.5.

$$\begin{aligned}
\text{Ab}(\pi_1(S_g)) &= \text{Ab}(\langle a_1, \ldots, a_{2g} \mid a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_2 g^{-1} = 1 \rangle) \\
&\cong \text{Ab}(\langle a_1, \ldots, a_{2g} \mid a_1 a_1^{-1} a_2 a_2^{-1} \cdots a_{2g-1} a_{2g-1}^{-1} a_{2g} a_2 g^{-1} = 1 \rangle)
\end{aligned}$$

This relation just becomes $1 = 1$, so we can ignore it.

$$\begin{aligned}
&= \text{Ab}(\langle a_1, \ldots, a_{2g} \rangle) \\
&= \mathbb{Z}^{2g}.
\end{aligned}$$

So $\pi_1(S_g)$ for different g are different, as after abelianization, the $\text{Ab}(\pi_1(S_g))$ are not isomorphic for different g.

Example 1.6.

$$\text{Ab}(\pi_1(N_g)) = \text{Ab}(\langle a_1, \ldots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 = 1 \rangle) \cong \mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z},$$

where if $\mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z} = \text{Ab}(\langle b_1, \ldots, b_g \mid b_g^2 = 1 \rangle)$, then the isomorphism sends $b_i \mapsto a_i$ for $i = 1, \ldots, g-1$ and $b_g \mapsto a_1 a_2 \cdots a_g$ (check this yourself).

Since $\text{Ab}(\pi_1)$ is distinct for every surface in our list, we conclude that no two of $S^2, S_1, S_2, \ldots, N_1, N_2, \ldots$ are homeomorphic. So we have proved the Poincarè conjecture for $n = 2$!

1.2 Miscellaneous topics in algebraic topology

The rest of this lecture is non-testable material but is included for interest.

1.2.1 Euler characteristic and orientibility

Here are two more things about surfaces:

1. Euler characteristics: $\chi(S) =$ "# vertices" - "# edges" + "# polygons" in a cellular decomposition. Check that all operations won't change this invariant.

2. orientibility: does a Möbius band embed in your surface (chapter 7 in Armstrong)

Using these two ideas, we can classify surfaces without using fundamental groups.

1.2.2 Homology

Definition 1.2. If X is a path-connected topological space, the *first homology group* of X is $H_1(X) = \text{Ab}(\pi_1(X))$.

If X is not necessarily path-connected, $H_0(X) \cong \mathbb{Z}^{\#\text{path-components}}$.

1.2.3 Low and high dimensional topology

In general, we can classify the study of manifolds by their dimension:

- $n \leq 3$: low dimensional topology (not enough room to go wrong, not weird)

- $n = 4$: most weird things happen (enough room to go wrong, not enough room to fix them)

- $n \geq 5$: high dimensional topology[2] (enough room to go wrong, enough room to fix them)

1.2.4 Higher homotopy groups

Choose $1 \in S^1$ and $p \in X$. Then

$$\pi_1(X, p) = \{\text{homotopy classes rel } \{1\} \text{ of continuous maps } S^1 \to X \text{ s.t. } 1 \mapsto p\}.$$

Definition 1.3. Let $x_0 \in S^n$ and $p \in X$. The *n-th homotopy group* of X based at p is

$$\pi_n(X, p) = \{\text{homotopy classes rel } \{x_0\} \text{ of continuous maps } S^n \to X \text{ s.t. } 1 \mapsto p\}.$$

What is the group operation? First, let $S^n \cong B^n \setminus \partial B^n$ and $B^n \cong \underbrace{[0,1] \times \cdots \times [0,1]}_{I^n}$.

So a map $f : S^n \to X$ such that $f(x_0) = p$ can be thought of as a map

$$I^n \xrightarrow{\phi} B^n \xrightarrow{p} S^n \xrightarrow{f} X.$$

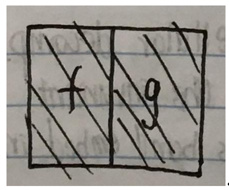
[2] This is Professor Conway's area of research.

We have a projection map $p : B^n \to S^n$, and we can let $x_0 = p(\partial B^n)$. The map $|phi : I^n B^n$ is a homeomorphism.

Now, given $f, g \in \pi_n(X, p)$, let $f \cdot g \in \pi_n(X, p)$ be

$$(f \cdot g)(x_1, \ldots, x_n) = \begin{cases} f(2x_1, x_2, \ldots, x_n) & x_1 \in [0, 1/2] \\ g(2x_1 - 1, x_2, \ldots, x_n) & x_1 \in (1/2, 1] \end{cases}$$

Example 1.7. Let $n = 2$. Then this looks like



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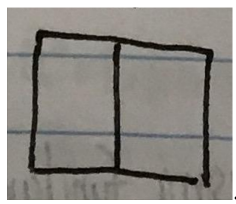
and $f \cdot g$ evaluates to p on the set



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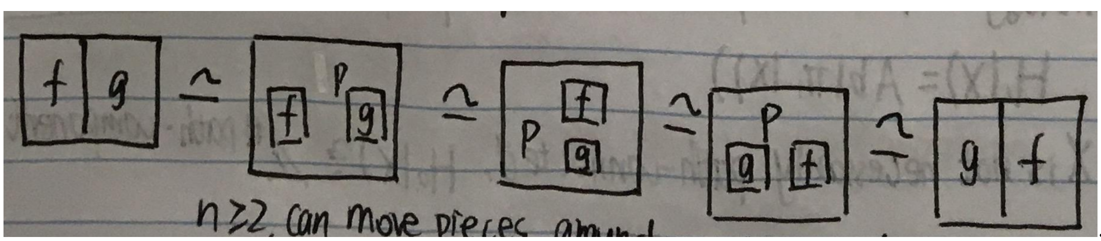
Theorem 1.2. *$(\pi_n(X, p), \cdot)$ is an Abelian group for all $n \geq 2$.*

Proof. Here is an intuitive sketch of why this is true. For $n \geq 2$, we can moe pieces around in a different direction.



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Theorem 1.3. *Paths from p to q induce isomorphisms $\pi_n(X, p)$ to $\pi_n(X, q)$, so if X is path-connected, we can write $\pi_n(X)$.*

Theorem 1.4. *For all $n > 1$, $\pi_n(S^1) \cong 1$.*

Proof. Here is the idea. Let $g : S^n \to S^1$. Find the lift $\tilde{g}$ of g. $\mathbb{R}$ is contractible, so $\tilde{g}$ is null homotopic. So g is, as well. ☐

Theorem 1.5. *For all $i < n$, $\pi_i(S^n) \cong 1$.*

Proof. Here is the idea. Show that any $g : S^i \to S^n$ is homotopic to $h : S^i \to S^n$ and $S^n \setminus h(S^i) \neq \varnothing$. Choose q in the complement. Then h is really a map $S^i \to S^n \setminus \{q\} \cong \mathbb{R}^n$. $\mathbb{R}^n$ is contractible, so h (and hence g) is null homotopic. □

Theorem 1.6. *$\pi_n(S^n) \cong \mathbb{Z}$ and is generated by $[\mathrm{id}_{S^n}]$.*

We can use this ro prove Brouwer's fixed point theorem in all dimensions. Homology is an easier way to do so.

What about $\pi_n(S^k)$ for $n > k > 1$? This is HARD. For the last 60 years, algebraic topologies have tried to solve this; now people are bored.

Example 1.8.

$$\pi_3(S^2) \cong \mathbb{Z},$$

$$\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z},$$

$$\pi_{14}(S^4) \cong \mathbb{Z}/120\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

There seems to be no general formula, but there exist techniques and subtle patterns, such as $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$.